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Certain Result involving Special Polynomials and Fractional Calculus

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Abstract: - The purpose of this paper is to consider a new generalization of the special polynomials. The combine use of integral transform and special polynomial provide a power full tool to deal with Fraction derivatives and integral.

Key words : Fractional Calculus, Special Polynomials, Generalized Gamma Function, Integral Transform.
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1. INTRODUCTION AND PRELIMINARIES.

The use of integral transforms to deal with fractional derivatives trace back to Riemann and Liouville [9,10]. The possibility of exploiting integral transforms in a wider context involving " exotic" operator has been discussed in references [4,5], where taking advantage form the definition of the generalized $\Gamma_{(b)}(\alpha)$ function [2,p.9].

$$\Gamma_b(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t-\frac{b}{t}}dt \qquad (1,1)$$

$$\left(\mathrm{R}(b)>0; b=0\ \ \mathrm{R}(\alpha)>0\right),$$

and

$$\Gamma_{ab}(\alpha)=a^\alpha\int_0^\infty t^{\alpha-1}e^{-at-\frac{b}{t}}dt \qquad (1.2)$$

$$\left(|a|+|b|\neq 0; b=0, \mathrm{R}(a)>0, \mathrm{R}(\alpha)>0\right).$$

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The function $\Phi(x,s,a)$ [1,7] has the integral representation

$$\Phi(x,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\left(1-xe^{-t}\right)^{-1}dt \qquad (1.3)$$

$$\left(\begin{array}{l}\mathrm{R}(a)>0; either\,|x|\leq 1, x=1\\ and\ \mathrm{R}(s)>0\ or\ x=1\mathrm{R}(s)>1\end{array}\right),$$

and class of function introduced in [7]

$$\Theta_\mu^\lambda\left(x,s,a,b\right)=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-at-\frac{b}{t^\lambda}}\left(1-xe^{-t}\right)^{-\mu}dt$$

$$\left(\begin{array}{l}\lambda>0,\mu\geq 1\mathrm{R}(a)>0,\mathrm{R}(b)>0; when\mathrm{R}(b)=0\\ the\ neither\,|x|\leq 1(x=1),\mathrm{R}(\alpha)>0, or\ x=1,\mathrm{R}(\alpha)\end{array}\right).$$

It has been shown that

$$\left[\frac{d}{dx}\right]_{(a,b)}!\,f(x)=a^\alpha\int_0^\infty e^{-at-\frac{b}{t^\lambda}}t^{\frac{d}{dx}}f(x)dt$$

$$=a^\alpha\int_0^\infty e^{-at-\frac{b}{t^\lambda}}f(x+\ln(t))dt \qquad (1.5)$$

An analogous result can be exploited for more complicated operator, be recalling, indeed , that

$$e^{\lambda x\left(\frac{d}{dx}\right)}f(x)=f(e^\lambda x)\ . \qquad (1.6)$$

We find

$$\left[\frac{d}{dx}\right]_{(a,b)}! f(x) = a^{\alpha}\int_0^{\infty} e^{-at-\frac{b}{t^{\lambda}}} t^{x\frac{d}{dx}} f(x)dt$$

$$= a^{\alpha}\int_0^{\infty} e^{-at-\frac{b}{t^{\lambda}}} f(xt)dt. \tag{1.7}$$

Furthermore noting form (1.2)

$$a^{-\alpha} = \frac{1}{\Gamma_{(ab)}(\alpha)}\int_0^{\infty} e^{-at-\frac{b}{t}} t^{\alpha-1}dt. \tag{1.8}$$

We can also conclude that

$$\left(\frac{d}{dx}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} t^{\alpha-1} e^{-\frac{b}{t^{\lambda}}} e^{-t\frac{d}{dx}} f(x)dt$$

$$= \frac{1}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} t^{\alpha-1} e^{-\frac{b}{t^{\lambda}}} f(x-t)dt, \tag{1.9}$$

or

$$\left(x\frac{d}{dx}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} t^{\alpha-1} e^{-\frac{b}{t^{\lambda}}} f(e^{-t}x)dt. \tag{1.10}$$

This last result becomes more interesting by noting that its relevance to the $\Theta_{\mu}^{\lambda}(x,s,a,b)$, $\Phi(x,s,a)$ and $\varsigma(\nu)$ functions defined in [7].

It is indeed readily understood that

$$\left(x\frac{d}{dx}\right)_{(b)}^{-\alpha}\left[\frac{1}{1-x}-1\right]^{\mu} =$$

$$\frac{1}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} t^{\alpha-1} e^{-\frac{b}{t^{\lambda}}}\left[\frac{1}{1-e^{-t}x}-1\right]^{\mu} dt$$

$$= \frac{x^{\mu}}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} e^{-\mu t-\frac{b}{t^{\lambda}}} t^{\alpha-1}\left[1-xe^{-t}\right]^{\mu} dt$$

$$= \frac{\Gamma(\alpha)}{\Gamma_{(b)}(\alpha)} x^{\mu}\Theta_{\mu}^{\lambda}(x,\alpha,\mu,b). \tag{1.11}$$

Special case of (1.11)

(i). $b=0, \lambda=\mu=1$, then

$$\left(x\frac{d}{dx}\right)^{-\alpha}\left[\frac{1}{1-x}-1\right] = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} x\,\Theta_1^1(x,\alpha,1)$$

$$= x\,\Phi(x,\alpha,1). \tag{1.12}$$

Where $\Phi(x,\alpha,1)$ is the generalized zeta function defined by (1.3) [1,7].

Case (ii) When $b=0$,

$$\left(x\frac{d}{dx}\right)^{-\alpha}\left[\frac{1}{1-x}-1\right]^{\mu} = x^{\mu}\Theta_{\mu}^{\lambda}(x,\alpha,\mu)$$

$$= x^{\mu}\Phi_{\mu}^{*}(x,\alpha,\mu), \tag{1.13}$$

which was studied recently by Goyal and Laddha [11].

The series representation of $\Phi_{\mu}^{*}(x,\alpha,\mu)$ is

$$\Phi_{\mu}^{*}(x,\alpha,\mu) = \sum_{n=0}^{\infty}\frac{(\mu)_n x^n}{(\mu+n)^{\alpha} n!} \tag{1.14}$$

$$\left(\mu \ge 1, \mathrm{R}(\alpha) > 0, |x \le 1|\right).$$

After clarifying that the use of meaningless operational form may have non trivial consequence if placed in a proper frame work .We will discuss further integral transform consequence of equation (1.8) , we find that

$$\left(\gamma - \frac{\partial}{\partial x}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{\Gamma_{(b)}(\alpha)} \int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} e^{t\frac{\partial}{\partial x}} f(x)dt$$

$$= \frac{1}{\Gamma_{(b)}(\alpha)} \int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} f(x+t)dt \,, \tag{1.15}$$

or

$$\left(\gamma - x\frac{\partial}{\partial x}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{\Gamma_{(b)}(\alpha)} \int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} f(e^{-t}x)dt \,. \tag{1.16}$$

Let us now consider the case involving second order derivatives namely

$$\left(\gamma - \frac{\partial^2}{\partial x^2}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{\Gamma_{(b)}(\alpha)} \int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} e^{t\frac{\partial^2}{\partial x^2}} f(x)dt \,. \tag{1.17}$$

The action of the second of the exponential operator containing the second order derivative can be specified by mean of the Gauss transform

$$e^{t\left(\frac{\partial^2}{\partial x^2}\right)} f(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)d\xi \,, \tag{1.18}$$

so that

$$\left(\gamma - \frac{\partial^2}{\partial x^2}\right)_{(b)}^{-\alpha} f(x) = \frac{1}{2\Gamma_{(b)}(\alpha)} \int_0^\infty \frac{e^{-\gamma t - \frac{b}{t^\lambda}}}{\sqrt{\pi t}} t^{\alpha-1} \left[\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4t}} f(\xi)d\xi\right] dt \,. \tag{1.19}$$

In this introductory section we have state that by combining the properties of exponential operator and suitable integral representation.

In this paper we will show how families of special polynomial may provide a powerful complement to the theory of fractional operators.

2. INTEGRAL TRANSFORM AND SPECIAL POLYNOMIALS.

It is well known that the polynomial [12]

$$H_n^{(2)}(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \,, \tag{2.1}$$

can be viewed as Gauss Transform of the ordinary monomial x^n, we find indeed,

$$H_n^{(2)}(x,y) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4y}} \xi^n d\xi \,. \tag{2.2}$$

According to equation (1.18) and (2.2) , we obtain

$$\left(\gamma - y\frac{\partial^2}{\partial x^2}\right)_{(b)}^{-\alpha} x^n = \frac{1}{\Gamma_{(b)}(\alpha)} \int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} H_n^{(2)}(x, yt)dt \tag{2.3}$$

The transform on the R.H.S of equation (2.3) defines new family of polynomials, denoted by ${}_\alpha H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)$.

The relevant generating function is easily obtained from their operational definition.

$$\left(\gamma - y\frac{\partial^2}{\partial x^2}\right)_{(b)}^{-\alpha} x^n = {}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma) \quad (2.4)$$

and the series representation

$$\left(\gamma - y\frac{\partial^2}{\partial x^2}\right)_{(b)}^{-\alpha} x^n = \frac{n!}{\Gamma_{(b)}(\alpha)} \times$$

$$\sum_{k=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-b)^k \Gamma(\alpha + r - \lambda k)\gamma^{\lambda k - \alpha - r} y^r x^{n-2r}}{k!r!(n-2r)!} \quad (2.5)$$

Special case of (2.5)

When $b = 0$,

$${}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma) = {}_{\alpha}H_{n}^{(2)}(x,y;\gamma),$$

the polynomial ${}_{\alpha}H_{n}^{(2)}(x,y;\gamma)$ was discuss by G.Dattoli and P.E. Ricci [4].

The properties of this new family of polynomial are fairly easy to obtain and we reported some properties hear.

From definition (2.5) we find .e.g.

$$\frac{\partial}{\partial x}\left({}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)\right) = n\left({}_{\alpha}H_{n-1}^{(2)}(x,y,b;\gamma)\right) \quad (2.6)$$

$$\frac{\partial}{\partial y}\left({}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)\right) =$$
$$n(n-1)\left({}_{\alpha}H_{n-2}^{(\infty,2)}(x,y,b;\gamma)\right) \quad (2.7)$$

$$\frac{\partial}{\partial b}\left({}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)\right) = \left({}_{\alpha-\lambda}H_{n}^{(\infty,2)}(x,y,b;\gamma)\right) \quad (2.8)$$

$$\frac{\partial}{\partial \gamma}\left({}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)\right) =$$
$$b\left(\frac{\Gamma_{(b)}(\alpha+1)}{\Gamma_{(b)}(\alpha)} - \alpha\right)\left({}_{\alpha+1}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)\right) \quad (2.9)$$

and

$$\left(\gamma - y\frac{\partial^2}{\partial x^2}\right)^{1-\alpha} x^n = \frac{\alpha}{\gamma}\frac{\Gamma_{(b)}(\alpha)}{\Gamma_{(b)}(\alpha-1)}\,{}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma) +$$
$$\frac{n(n-1)}{\gamma}\frac{\Gamma_{(b)}(\alpha+1)}{\Gamma_{(b)}(\alpha-1)}\,y\,{}_{\alpha+1}H_{(n-2,\lambda)}^{(\infty,2)}(x,y,b;\gamma) -$$
$$\frac{\Gamma_{(b)}(\alpha-\lambda)}{\Gamma_{(b)}(\alpha-1)}\frac{b}{\gamma^{\lambda-1}}\,{}_{\alpha-\lambda}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma) \quad (2.10)$$

Further result will be discussed in the following section.

3, **Extension and Concluding Remarks**

We have already remarked that the polynomials ${}_{\alpha}H_{(n,\lambda)}^{(\infty,2)}(x,y,b;\gamma)$ can be recognized as special form of truncated polynomials we will further discuss identifications.

According to the discussion of the previous section, the following identity can easily be proved:

$$\left(\gamma - y\frac{\partial}{\partial x}\right)_{(b)}^{-\alpha} x^n = \frac{1}{\Gamma_{(b)}(\alpha)}\int_0^{\infty} e^{-\gamma t - \frac{b}{t^{\lambda}}} t^{\alpha-1}\left(x+yt\right)^n dt$$

$$= {}_{\alpha}H_{(n,\lambda)}^{(\infty,1)}(x,y,b;\gamma) \quad (3.1)$$

The properties of the polynomials ${}_{\alpha}H_{(n,\lambda)}^{(\infty,1)}(x,y,b;\gamma)$ can also be easily recovered. It is However worth stressing their link with the so called Bessel Polynomials [9]. By setting

$b=0, \alpha=n+1, x=1, y=\frac{z}{2}$, we can make the identification

$$ {}_{n+1}H_n^{(1)}\left(1, \frac{z}{2}, 1\right) = y_n(z) , \tag{3.2}$$

In this concluding section we find it worthwhile to mention briefly hear a multivariable extension of the class of function ${}_{\alpha}H_{(n,\lambda)}^{(\infty,1)}(x,y,b;\gamma)$ .This multivariable polynomials

${}_{\alpha}H_{(n,\lambda)}^{(\infty,1)}\left(\{x\}_1^m, y, b; \gamma\right)$ can be defined by

$$\left(\gamma - \sum_{s=2}^{m} x_s \frac{\partial^s}{\partial x_1^s}\right)_{(b)}^{-\alpha} x_1^n = \frac{1}{\Gamma_{(b)}(\alpha)} \times$$
$$\int_0^\infty e^{-\gamma t - \frac{b}{t^\lambda}} t^{\alpha-1} H_n^{(m)}(x_1, x_2 t, x_3 t, \ldots, x_m t) dt$$
$$= {}_{\alpha}H_{(n,\lambda)}^{(\infty,1)}\left(\{x\}_1^m, y, b; \gamma\right) \tag{3.3}$$

where

$$\{x\}_1^m = x_1, x_2, \ldots. x_m.$$

Further comment on the properties of the above function and on their usefulness in application will be presented elsewhere.

REFERENCES

1. Andrews L.C., *Special Function for Engineering and Applied Mathematicians*, Macmillan ,NewYork,1985.
2. Chaudahry M.Aslam and Zubair Syed M. , *On a Class of Incomplete Gamma Functions with Application*, Chapman & Hall/CRC (Boca Raton/ London/ New York/Washington D.C.)2000.
3. Dattoli G., Cesarano C. and sacchetti D., A note on truncated polynomials, Applied Math and Computational (appear).
4. Dattoli G., Ricci P.E. and Cesarano C., Fractional derivatives: Integral representation and generalized polynomials, J. Coner Applic. Math (appear).
5. Dattoli G., Ricci P.E., Cesarano C. and Vaxquez L., Special polynomials and fraction calculus, Math and Comp.Modelling 37(2003),729-733.
6. Oldham H.and Spainer N., *The Fractional Calculus*, Academic Press ,San Diego, CA, 1974.
7. Raina R.K. and Chhajed P.K., Certain results involving a class of functions associated with the hurwitz zeta function. Acta Math.Univ.Comeniance Vol. LXXIII,(2004),89-100.
8. Raina R.K. and Nahar T.S., A note on certain class of function related to hurwitz zeta function and Lambert transform, tamkang J.Math 31(2000),49-55.
9. Rainvile E.D., *Special Functions*, Macmillian, New York, 1960.
10. Widder D.V., *An Introduction to Transform Theory*, Academic Press, New York 1971.